The lift on a tank-treading ellipsoidal cell in a shear flow

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Abstract

The lift on a strongly non-spherical vesicle in a bounded shear flow, is studied in the case the membrane moves with a velocity, which is a linear function of the coordinates. The magnitude of the induced drift is calculated as a function of the axes lengths, of the distance from the wall, and of the ratio of the cell to the solvent viscosity. It appears that the main mechanism for lift, in the presence of tank-treading motions, is the fixed orientation of the vesicle with respect to the flow. Tank-treading vesicles in suspensions, flowing through narrow gaps under small Reynolds number conditions: $Re = Lv/\nu < 1$ (with L the gap width, v the flow velocity and ν the viscosity) migrate away from the walls, with a velocity that is $\mathcal{O}(Re^{-1})$ larger than predicted by inertia.

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An important property of suspensions, both in Couette and channel flow conditions, is the ability to change their effective viscosity, by concentration of the particles, near the centre of the flow [1, 2]. One of the most striking examples is the behavior of red cells in small blood vessels, where this phenomenon takes the name of Fahraeus-Lindqwist effect [3].

In the case of spherical particles, it is well known that the only mechanism, which is able to cause a transverse migration of particles, is the presence of inertial corrections to the Stokes equation, which describes the response of the solvent to the particles [4, 5, 6]. This is not true anymore, however, when spherical symmetry is lost. In this case, a particle can behave like a sort of wing and move in a direction different from that of the flow.

Very often, in particular when one deals, as in the case of red cells [7], with vesicles, or otherwise, with droplets of an unmiscible liquid, the particle shape is approximately that of an ellipsoid. If the ellipsoid is rigid, it will not maintain a fixed orientation, but will rotate as a whole in an unsteady fashion, carrying on a kind of flipping motion [8]. One suspects in this case, that spherical symmetry is in some way recovered and that Stokes dynamics is not able to produce by itself a transverse drift. Things change in the case of an object with a compliant structure, where tank-treading motions become possible and a fixed orientation ensues.

Recently, the behavior of a spheroidal vesicle, undergoing a tank-treading motion in a bounded shear flow, has been analysed using perturbation theory around the spherical particle case [9]. For a spherically symmetric geometry, a Green function for the Stokes equation is available and this equation can then be solved, for arbitrary boundary conditions on the surface of the cell.

For strongly non-spherical objects, however, the only tractable situation is that of the rigid ellipsoid, studied by Jeffery [8]. Keller and Skalak [10] showed that Jeffery's analysis could be generalised to situations, in which the boundary condition on the cell surface, is that the velocity be a generic linear function of the coordinates. The special case they considered, with a cell assumed to have an ellipsoidal shape, and with the points on its surface moving with constant angular velocity, provided a useful approximation for the locally area preserving motion of a red cell membrane.

Of course, this approach solves only a part of the problem, leaving out the determination of the shape and internal motions of a cell in an external shear. A self-consistent description, like the one contained e.g. in [11, 12, 13], is however not really necessary here, at least for large distances from the walls, in which case, no large deformations are expected.

In this paper, the theory of Jeffery, and of Keller and Skalak, is used to calculate the lift on a vesicle, undergoing the special kind of tank-treading motion studied in [10]. Analysis carried on in [9], showed in the perturbative regime, that different kinds of tank-treading motions produce lifts, which are essentially of the same order of magnitude. (More precisely, the case considered in [10] gave a lift, which was approximately twice as large as that of a cell, with a membrane moving in an exactly area preserving fashion).

As in [9], the analysis is confined to a range of distances from the wall, such that a calculation of the cell-wall interaction, based on the method of images is possible, and that the far field expression for the velocity disturbance from the cell can be used. Extrapolating the results to the region near the wall, it appears that a strongly non-spherical tank-treading vesicle, migrates towards the centre of the fluid, with a speed of the same order of the difference of velocity, due to the external shear, that exists across its body. This speed is typically much larger than the one produced by inertia.

Consider a cell, whose shape is that of an ellipsoid, with axes $a_2 \leq a_1 \leq a_3$, immersed in a shear flow, which, in an appropiate reference system $\{x_1, x_2, x_3\}$, translating with the ellipsoid and with origin at its centre, can be written in the form: $\bar{\mathbf{v}} = \kappa x_2 \mathbf{e}_3$. Indicate with $\{x'_1, x'_2, x'_3\}$ the reference system in which the equation for the ellipsoid surface can be written in the diagonal form:

$$\frac{x_1'^{\,2}}{a_1^2} + \frac{x_2'^{\,2}}{a_2^2} + \frac{x_3'^{\,2}}{a_3^2} = 1. \tag{1}$$

In a state of tank-treading motion, the cell will keep a fixed orientation, with $x'_1 \equiv x_1$, and x'_3

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forming an angle $\bar{\theta}$ with respect to x_3, with $0 < \bar{\theta} < \pi/4$ for $\kappa > 0$ (see Fig. 1). For vanishing internal viscosity, or in the almost spherical case, $\bar{\theta} \to \pi/4$, corresponding to the strain part of the shear flow dominating the dynamics and forcing the ellipsoid and the strain axes to get aligned. In the other limit, $\bar{\theta} = 0$ signals the crossover to the vorticity dominated, flipping motion regime.
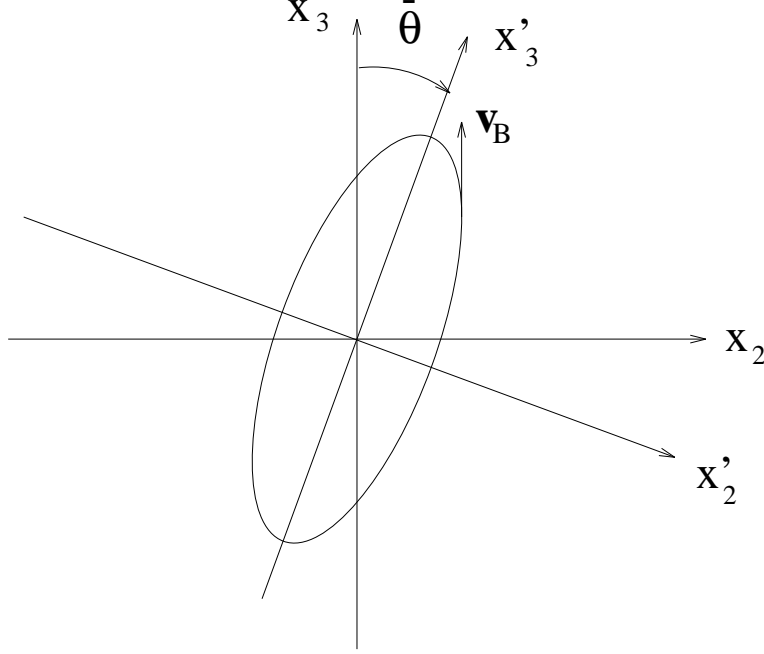

Figure 1: Orientation of a tank-treading cell in a plane shear flow $\bar{\mathbf{v}} = \kappa x_2 \mathbf{e}_3$.

The constant angular velocity tank-treading motion, that is taken into exam, is described by the equation for the velocity at the membrane [10]:

$$\mathbf{v}_B = \omega \kappa \left(-\frac{a_2}{a_3} x_3' \mathbf{e}_2' + \frac{a_3}{a_2} x_2' \mathbf{e}_3' \right). \tag{2}$$

This kind of motion can easily be shown not to be area-preserving by an amount of the order of $\log a_3/a_2$; however, Eqn. (2) goes beyond providing an expression for $\mathbf{v}_B$ which allows analytical calculations. Recently, Kraus et Al. [11] have obtained semi-analytical expressions, for $\bar{\theta}$ and ω in terms of $\mathbf{v}_B(x_1 = 0)$, and for $\mathbf{v}_B(x_1 = 0)$ itself, keeping into account all the couplings between the membrane and the fluid. It turns out that, in the spheroidal limit, the expression given by Eqn. (2) for $v_B(x_1 = 0)$, (and then also for $\bar{\theta}$ and ω) is equal to the one obtained in [11].

The boundary condition for the velocity perturbation $\mathbf{v}$ produced by the cell, is given by $\mathbf{v} = \bar{\mathbf{v}} - \mathbf{v}_B$ on the cell surface, and can be written in the form:

$$v_1' = 0, \qquad v_2' = \kappa(bx_2' + (f - \xi)x_3'), \quad v_3' = \kappa(-bx_3' + (f + \xi)x_2'), \tag{3}$$

where:

$$b = \frac{\sin 2\bar{\theta}}{2}, \quad \text{and} \quad f = \frac{1}{2}\left[\cos 2\bar{\theta} + \omega\left(\frac{a_2}{a_3} - \frac{a_3}{a_2} \right) \right] \tag{4}$$

are the components of the strain matrix, while

$$\xi = \frac{1}{2}\left[1 - \omega\left(\frac{a_2}{a_3} + \frac{a_3}{a_2} \right) \right], \tag{5}$$

is the vorticity of $\mathbf{v}$. Jeffery's theory [8] gives the following far field expression for the velocity disturbance:

$$\mathbf{v} \simeq -\frac{4\kappa R^3 \mathbf{x}}{x^5}\Phi(\mathbf{x}) \tag{6}$$

where $\Phi = A{x'_1}^2 + B{x'_2}^2 + C{x'_3}^2 + 2F x'_2 x'_3$ and $R = (a_1 a_2 a_3)^{\frac{1}{3}}$. In the unprimed laboratory frame:

$$\Phi = Ax_1^2 + \left(\frac{B-C}{2}\cos 2\bar{\theta} + \frac{B+C}{2} - F\sin 2\bar{\theta}\right)x_2^2$$

$$+\left(-\frac{B-C}{2}\cos 2\bar{\theta} + \frac{B+C}{2} + F\sin 2\bar{\theta}\right)x_3^2$$

$$+[(B-C)\sin 2\bar{\theta} + 2F\cos 2\bar{\theta}]x_2 x_3 \tag{7}$$

The coefficients A, B, C and F are given by the expressions [8]:

$$A = \frac{(\alpha''_3 - \alpha''_2)b}{Z}; \qquad B = \frac{(2\alpha''_2 + \alpha''_3)b}{Z}; \qquad C = -\frac{(\alpha''_2 + 2\alpha''_3)b}{Z};$$

$$F = \frac{f}{2\alpha'_1(\hat{a}_2^2 + \hat{a}_3^2)} \quad \text{where:} \quad Z = 6(\alpha''_1\alpha''_2 + \alpha''_2\alpha''_3 + \alpha''_3\alpha''_1), \tag{8}$$

where $\hat{a}_i = R^{-1}a_i$, and the α-coefficients, which depend solely on the geometry of the cell, are given by:

$$\alpha'_1 = \int_0^\infty \frac{d\lambda}{(\hat{a}_2^2 + \lambda)(\hat{a}_3^2 + \lambda)\Delta}, \qquad \alpha'_2 = \int_0^\infty \frac{d\lambda}{(\hat{a}_3^2 + \lambda)(\hat{a}_1^2 + \lambda)\Delta}, \cdots$$

and:

$$\alpha''_1 = \int_0^\infty \frac{\lambda d\lambda}{(\hat{a}_2^2 + \lambda)(\hat{a}_3^2 + \lambda)\Delta}, \cdots \tag{9}$$

where

$$\Delta = \sqrt{(\hat{a}_1^2 + \lambda)(\hat{a}_2^2 + \lambda)(\hat{a}_3^2 + \lambda)}. \tag{10}$$

Writing $a_1 = a_2 = (1-\epsilon)^{\frac{1}{2}}R$, $a_3 = (1+\epsilon)^{\frac{1}{2}}R$ and taking the limit $\epsilon \to 0$, one obtains:

$$\Phi \simeq \sin 2\bar{\theta}[-2\hat{A}x_1^2 + (\hat{A} - \tilde{F})x_2^2 + (\hat{A} + \tilde{F})x_3^2] + 2(\hat{F} + \tilde{F}\cos 2\bar{\theta})x_2 x_3, \tag{11}$$

where:

$$\tilde{F} = -\frac{5\epsilon}{16} \qquad \hat{A} = -\frac{5\epsilon}{112}, \quad \text{and} \quad \hat{F} = \frac{5}{8}\left(1 - \frac{5\epsilon}{14}\right). \tag{12}$$

Upon substitution into Eqn. (6), Eqns. (11-12) give the expression for the velocity disturbance by a spheroidal particle, obtained in [9].

The orientation angle $\bar{\theta}$ and the tank-treading frequency ω were calculated in [10] by Keller and Skalak, imposing angular momentum conservation, and requiring that viscous dissipation be equal to the work done by $\bar{\mathbf{v}}$ on the system:

$$\cos 2\bar{\theta} = \frac{z_1 z_3}{4}\left[\frac{z_1^2}{2} - \frac{1}{(1 - \epsilon_\nu)z_2 - 2}\right]^{-1}, \tag{13}$$

$$\omega = -\frac{\cos 2\bar{\theta}}{z_1((1 - \epsilon_\nu)z_2 - 2)}. \tag{14}$$

In the equations above:

$$z_1 = \frac{1}{2}\left(\frac{a_3}{a_2} - \frac{a_2}{a_3}\right), \qquad z_2 = \alpha'_1(\hat{a}_3^2 + \hat{a}_2^2), \qquad z_3 = \frac{a_3}{a_2} + \frac{a_2}{a_3}. \tag{15}$$

and ϵ_ν is the ratio of the inner to outer viscosity, with the inner viscosity being in general a weighed average of the viscosity of the membrane and that of the fluid inside [10]. The situation in which the magnitude of the RHS of Eqn. (13) is greater than one (as it happens for instance in the rigid particle limit $\epsilon_\nu \to \infty$) corresponds to the flipping motion regime.

The derivation of Eqns. (13-15) becomes very simple when $\epsilon_\nu = 1$ and the membrane viscosity is equal to zero, which corresponds to the case considered in [11]. It is necessary to know the expression of the velocity inside the cell, but it is easy to show that this is given by the same Eqn. (2) describing the membrane motion (see e.g. [9] and the expression for the inner solutions to the Stokes equation derived there). Imposing that no energy be dissipated in the membrane, forces the vorticity to be continuous across the cell surface, leading to the condition: $\xi = 0$, while, requiring that no torque be present in Eqn. (3), leads to the strain being aligned with the ellipsoid axes: $f = 0$. These two conditions imply: $v_2' = \kappa b x_2'$ and $v_3' = -\kappa b x_3'$, and using Eqns. (4-5), the result of Eqns. (13-15) for $\epsilon_\nu = 1$ is recovered.

A plane wall, placed perpendicular to the x_2-axis, at a distance l from the cell, causes a disturbance $\mathbf{v}^\mathrm{I}$ in the velocity field, which, at the cell centre results in a net drift away from the wall. This drift can be calculated, if the wall is sufficiently far away, by the image method, imposing the no-slip boundary condition at the wall: $\mathbf{v}^\mathrm{I} = -\mathbf{v}$. The calculation, whose details are illustrated in [9], is carried on introducing a representation in terms of potentials: $\mathbf{v}^\mathrm{I} = \nabla\phi + \nabla \times \mathbf{A}$, $\nabla \cdot \mathbf{A} = 0$, and Fourier transforming in x_1 and x_3. The potentials satisfying the Stokes equation and incompressibility for $\mathbf{v}^\mathrm{I}$ are:

$$\mathbf{A_k}(x_2) = (x_2 - l)\tilde{\mathbf{A}}_\mathbf{k} \exp(k(x_2 - l)), \qquad \phi_\mathbf{k}(x_2) = \tilde{\phi}_\mathbf{k} \exp(k(x_2 - l)). \tag{16}$$

where $f_\mathbf{k}(x_2) = \int \mathrm{d}x_1 \mathrm{d}x_3 \exp(-i\mathbf{k} \cdot \mathbf{x}) f(\mathbf{x})$, $f = \{\phi, \mathbf{A}\}$ and $\mathbf{k} = \{k_1, k_3\}$. Imposing the boundary condition on $\mathbf{v}_\mathbf{k}^\mathrm{I}(x_2)$, by means of Eqns. (6-7), and inverse Fourier transforming the result, the large distance expression for the drift $v^\mathrm{L} = -v_2^\mathrm{I}(0)$ is obtained:

$$v^\mathrm{L} = \frac{U\kappa R^3}{l^2} \tag{17}$$

where $U = U(\epsilon_\nu, r_1, r_2)$, with $r_i = a_i/a_3$, is twice the coefficient of the x_2^2 term entering Φ, as given in Eqn. (7):

$$U = (B - C)\cos 2\bar{\theta} + (B + C) - 2F\sin 2\bar{\theta}. \tag{18}$$

In the almost spherical limit, Eqns. (11-12) lead to the result of [9] $U \simeq \frac{15}{28}\epsilon \sin 2\bar{\theta}$, with $\bar{\theta} \simeq \pi/4$. The quantity U, which is of order 1 for large departures from sphericity, is a dimensionless velocity, normalised to κR, and gives the magnitude of the drift close to the wall.

Given Eqns. (6-9) and (13-15), the functional form of $U(\epsilon_\nu, r_1, r_2)$ can easily be explored. As it was to be expected, there is a strong correspondence between $U(\epsilon_\nu, r_1, r_2)$ and $\bar{\theta}(\epsilon_\nu, r_1, r_2)$; in particular: $U(\bar{\theta} \to 0) = 0$, which corresponds to the lift vanishing at the transition to flipping motion.

If both parameters ϵ_ν and r_2 are kept fixed, the strongest lift is achieved when $r_1 = r_2$ (see Fig. 2), which parallels the effect, observed in [10], that tank-treading motions are more robust in prolate rather than oblate ellipsoids. This is not due, however, to the orientation angle $\bar{\theta}$, being the largest in one of the two cases. It appears instead that $\bar{\theta}$ is independent of r_1: $\bar{\theta} = \bar{\theta}(\epsilon_\nu, r_2)$.

The strongest variations in U takes place, when either the viscosity ratio ϵ_ν or the non-sphericity (which can be parametrized by r_2^{-1}) are large. This corresponds to the intuition that strongly non-spherical, stiff cells, prefer to stay in a flipping motion state. In Fig. 3, as an example of strongly non spherical object, the red cell model studied in [10] is considered, and compared with the case of a spheroid. The red cell, both in a stretched and an unstretched state, make a transition to flipping motion roughly at $\epsilon_\nu = 3$, while the lift on a spheroid, remains almost constant up to $\epsilon_\nu \sim 10$. For fixed ϵ_ν, the lift grows with the degree of non-sphericity, from $U = 0$ at $\epsilon = 0$, and reaching a maximum at a critical $\bar{\epsilon} = \bar{\epsilon}(\epsilon_\nu)$, when $r_1 = r_2$. For $\epsilon > \bar{\epsilon}$, U decreases and reaches zero at the transition to flipping motion (see Fig. 4).
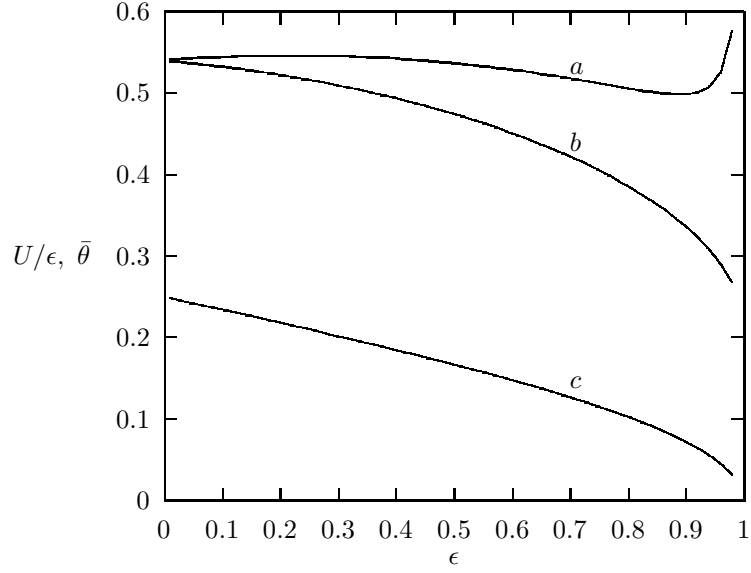
Figure 2: Plot of the drift U in function of the eccentricity $\epsilon = (\frac{1-r_2}{1+r_2})^{\frac{1}{2}}$, for $\epsilon_\nu = 1$, in the case of axisymmetric ellipsoids: (a) prolate $r_1 = r_2$; (b) oblate $r_1 = 1$. The orientation angle $\bar{\theta}$ (c) remains the same in the two cases.
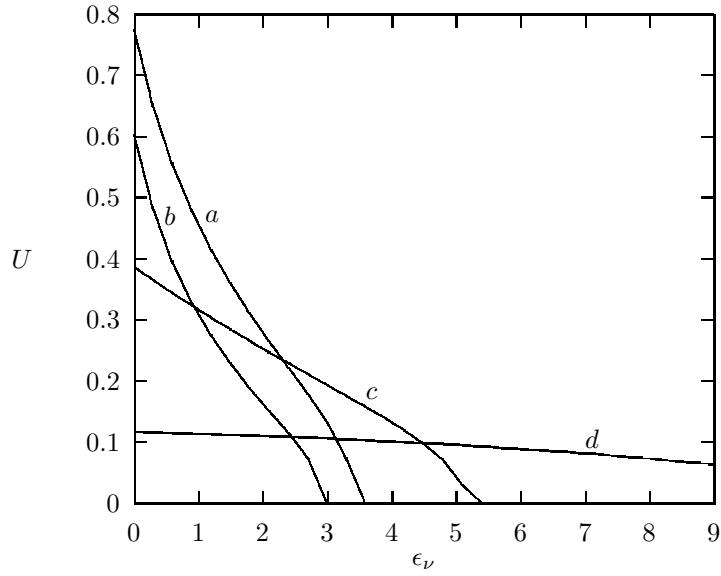

Figure 3: Dependence of U on the viscosity ratio ϵ_ν; red cell at rest: (a) $r_1 = 1$, $r_2 = 0.286$; (b) stretched red cell: $r_1 = r_2 = 0.213$; (c) a prolate ellipsoid: $r_1 = r_2 = 0.5$; an oblate spheroid: $r_1 = 1$; (d) $r_2 = 0.8$.
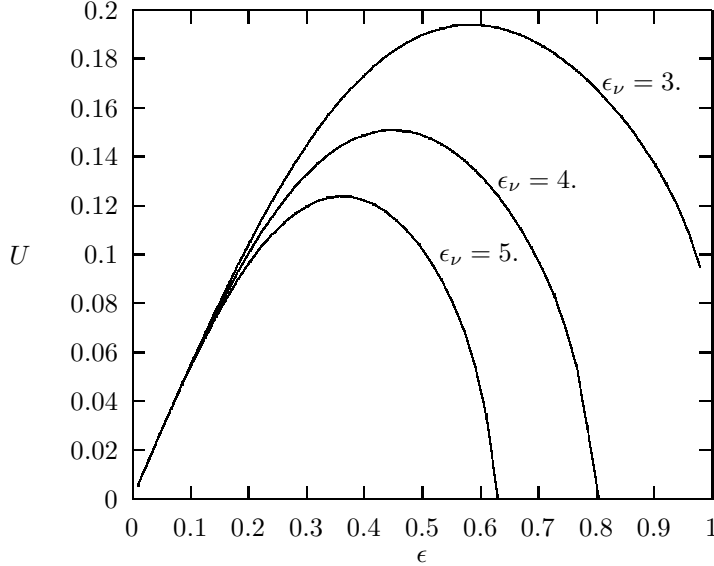
Figure 4: Dependence of U on ϵ, in the case of prolate ellipsoids, for different values of ϵ_ν. Notice the maxima moving towards $\epsilon = 0$ as the viscosity ratio is increased.

For ϵ_ν small, it appears from Fig. 2, that the variations in U/ϵ, that take place in the case of axisymmetric ellipsoids, are of the same order of those due to changes in the type of tank-treading motion. Thus, for small ϵ_ν, the linear theory in [9], keeps doing an acceptable job also for large values of ϵ. Notice however, that the linear theory remains unable to make predictions on $\bar{\theta}$, which instead, is kept fixed at $\bar{\theta} = \pi/4$.

In the case of the red cell model studied in [10], Fig. 5 illustrates the dependence of the lift on the degree of stretching and on ϵ_ν. It must be remembered that, in that case, the ratio ϵ_ν must account for the viscosity of the membrane as well as that of the hemoglobin inside [14]. However, for $\epsilon_\nu \simeq 1$, corresponding to red cells in a strongly viscous solvent, one sees that $U \simeq 0.3$, with not much variation when the cell is stretched by the shear. One can compare the prediction of Eqn. (17), with the contribution to lift from inertia, which is: $v^{\mathrm{L}} \sim \frac{f_0 \kappa^2 R^3}{\nu}$, with ν the viscosity of the suspension, and f_0 of the order of a few tenths [5]. Thus, if one has a suspension of red cells in a viscous solvent, forced to flow in a gap of thickness L, the condition for the lift on the red cells, to be dominated by inertia, becomes $\kappa L^2 > \nu$, i.e. a large Reynolds number condition for the flow in the gap.

From an experimental point of view, this means that the time $T \sim L/v^{\mathrm{L}}$ for cells to concentrate near the centre of a channel flow, as revealed e.g. through viscometric measurement, will obey different laws in the two regimes, so that $T = T(\kappa)$ must have a kink at $\bar{\kappa} \sim \nu L^{-2}$, with $T \propto \kappa$ for $\kappa < \bar{\kappa}$ and $T \propto \kappa^2$ for $\kappa > \bar{\kappa}$. These experiments are complicated however by the contribution of vesicle deformability to the suspension effective viscosity [15].

The goal of this research was to extend the analysis carried on in [9] to a non-perturbative regime. It appears that the drift produced by tank-treading motions approaches, close to the wall, the magnitude of the velocity difference at the particle scale κR. It must be stressed, that in this region, the approximations leading to Eqn. (17) break up, so that the expression for the drift this equation provides, has a value only as an order of magnitude estimate. For larger values of l/R, however, Eqn. (17) becomes exact. This furnishes then, a test for numerical solvers, once a boundary condition like that provided by Eqn. (2) is imposed. For $l \sim R$, a more quantitative analysis would require a full solution of the fluid equations, in the presence of both the cell and the wall. Due to the complicated geometry of the problem, a numerical approach would become
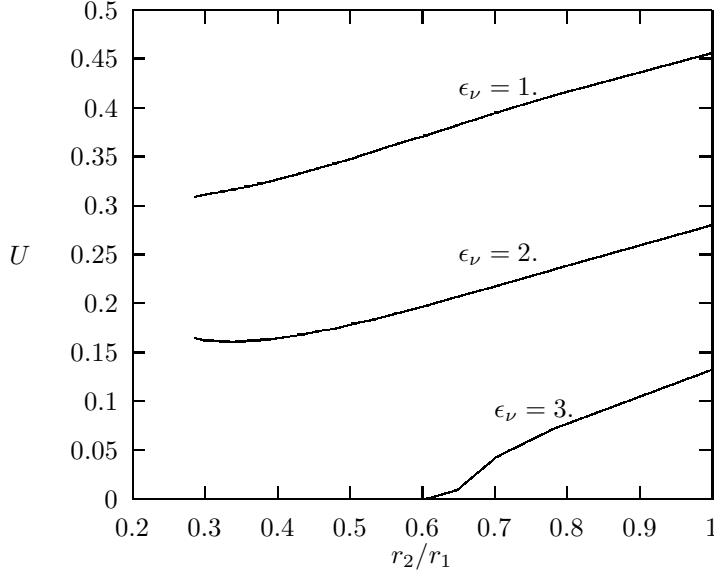
Figure 5: Values of the lift on a red cell in solvents of different viscosity, in function of the stretching: r_2/r_1.

necessary.

Looking at the problem from a different angle, analysis carried on in [9] suggested a rather strong dependence of the drift on the exact nature of the tank-treading motion, and similarly, on deviations from a strictly ellipsoidal cell shape. The magnitude of the drift appears to be strongly influenced, also by the viscosity of the membrane and that of the fluid inside the cell. Hence, considering both measurement difficulties, and how details in shape and membrane motion are affected by perturbations, an exact knowledge of the value of U close to the wall is perhaps not very interesting, and the use of a simple expression for $\mathbf{v}_B$, like the one provided by Eqn. (2), is appropiate.

As an aside, these results imply that, a cell capable of producing tank-treading motions, has at its disposal a propulsion mechanism, which, beyond obtaining most of the necessary energy from the external flow, can be controlled by small changes in cell shape. A similar mechanism for cell propulsion has been suggested in [16].

Despite these strong dependencies, one expects the production of lift from the fixed orientation and non-sphericity of the cell, to be in itself a robust phenomenon. In particular, the same effect should be present, as assumed earlier, even when the large distance condition is relaxed, and more important, when quadratic corrections in the shear are taken into account. (In fact, in Poiseuille and channel flow configurations, one would expect a stabilising effect on the cell orientation [17]).

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